Approximation of a Reaction-Diffusion Equation with a Nonlocal Term (Variational Problems and Related Topics)

Okada, Koji

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Approximation of a Reaction-Diffusion Equation with a Nonlocal Term

広島大学大学院理学研究科・数理分子生命理学専攻 岡田 浩嗣 (Koji Okada)
Department of Mathematical and Life Sciences,
Graduate School of Science, Hiroshima University

1 Introduction.

We consider a scalar bistable reaction-diffusion equation

(RD) $$\epsilon u_t = \epsilon^2 \Delta u + f(u) - v, \qquad t > 0,\ x \in \Omega,$$

under the Neumann boundary condition

(BC) $$\frac{\partial u}{\partial \mathbf{n}} = 0, \qquad t > 0,\ x \in \partial\Omega.$$

Here u is an order parameter while v an additional parameter (acting as inhibitors). Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 2$) and $\mathbf{n}$ stands for the outward unit normal vector on the boundary $\partial\Omega$. The nonlinear term f is assumed to be the negative derivative of a smooth double-well potential W: $f(u) = -W'(u)$. A typical example is $f(u) = u - u^3$. The parameter $\epsilon > 0$ is supposed to be very small, and we intend to study the problem above as the singular perturbation problem.

We will treat in this paper a situation in which the spacial average of the order parameter is preserved:

(PP) $$\frac{1}{|\Omega|} \int_\Omega u(t,x)\,dx \equiv m \quad \text{(constant)}, \qquad t \geq 0,$$

i.e., a case where v in (RD) is given by

(NL) $$v(\cdot) = \frac{1}{|\Omega|} \int_\Omega f(u(\cdot, x))\,dx.$$

When $\epsilon > 0$ is very small, the solution $u(t,x)$ of (RD) with an appropreate initial condition creates a sharp transition layer with width of $O(\epsilon)$ and it is expected to move according to some motion laws, called interface equations. Our purpose of this paper is (1) to derive interface equations from (RD); and (2) to investigate how solutions of interface equations evolve.

Remark 1. From a variational point of view, the equation (RD) is characterized as the $L^2(\Omega)$-gradient system for the energy functional of van der Waals type

$$E^\epsilon(u) := \int_\Omega \frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon} W(u)\,dx$$

subject to the constraint (PP), and the nonlocal term v is regarded as the Lagrange multiplier (see [2] for example).

2 Derivation of interface equations.

Throughout the remaining part of this paper, an interface is meant to be a smooth, closed, $N-1$ dimensional hypersurface emmbedded in $\Omega \subset \mathbb{R}^N$. We will derive some interface equations from (RD) by the method of matched asymptotic expansions (see [9] for more details).

2.1 Preliminaries.

We now present precise assumptions on f and prepare some notations for our problem.

(A1) The function f is C^∞ on $\mathbb{R}$ and the curve $f(u)-v=0$ consists of three sub-branches of solutions

$$\mathcal{C}^- = \{(u,v)\,|\,u = h^-(v),\, v \in I^- := (\underline{v}, \infty)\},$$
$$\mathcal{C}^+ = \{(u,v)\,|\,u = h^+(v),\, v \in I^+ := (-\infty, \overline{v})\},$$

and

$$\mathcal{C}^0 = \{(u,v)\,|\,u = h^0(v),\, v \in I^0 := I^- \cap I^+ = (\underline{v}, \overline{v})\},$$

satisfying $f'(h^\pm(v)) < 0$ (or equivalently $h^\pm_v(v) < 0$) on $I^\pm$.

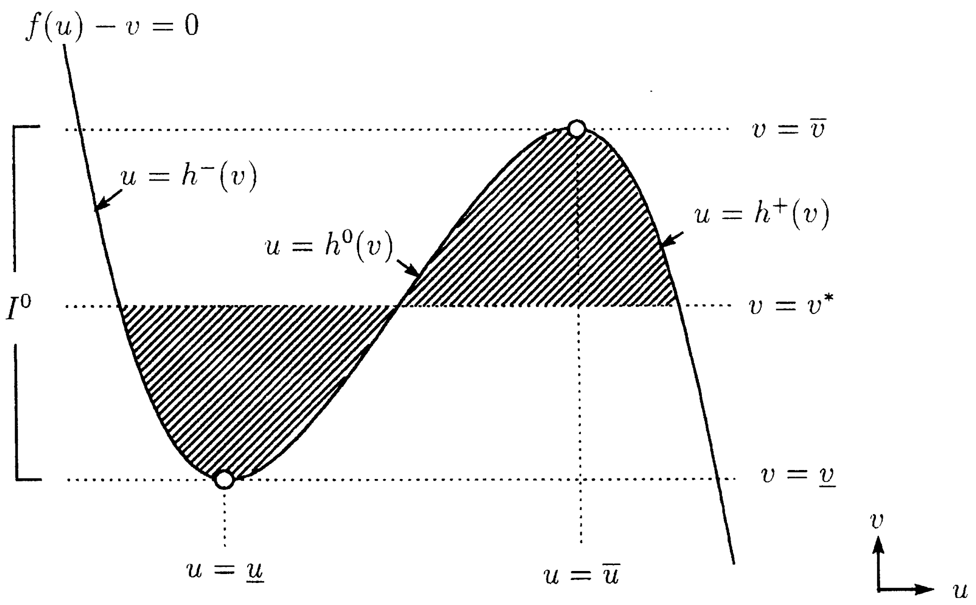
(A2) For each $v \in I^0$, it holds that $h^-(v) < h^0(v) < h^+(v)$.

(A3) For each $v \in I^0$, we define

$$\mathcal{S}(v) := \int_{h^-(v)}^{h^+(v)} f(u) - v\, du.$$

Then there exists a unique point $v^* \in I^0$ such that $\mathcal{S}(v^*) = 0$ and $\mathcal{S}'(v^*) < 0$.

Remark 2. We may regard the point $(h^0(v^*), v^*)$ as the origin $(0,0)$ by appropreate translations.



An unknown interface $\Gamma(t)$, which is to be determined, is expressed as a smooth embedding from a fixed $N-1$ dimensional reference manifold $\mathcal{M}$ to $\mathbb{R}^N$:

$$\gamma(t,\cdot):\mathcal{M}\to\Gamma(t)\subset\Omega,\qquad \mathcal{M}\ni y\mapsto x=\gamma(t,y)\in\Gamma(t). \tag{2.1}$$

Let $\Omega^{\pm}(t)$ be subregions (called bulk regions) in Ω decomposed by $\Gamma(t)$ such as

$$\Omega=\Omega^-(t)\cup\Gamma(t)\cup\Omega^+(t),$$

and $\nu(t,y)\in\mathbb{R}^N$ the unit normal vector on $\Gamma(t)$ at $x=\gamma(t,y)$ pointing into the interior of the bulk region $\Omega^+(t)$. In advance we standardize the parametrization as in (2.1) in such a way that $\gamma_t(t,y)$ is always parallel to $\nu(t,y)$ [3]. For sufficiently small $\delta>0$, a point x in a neighborhood $\{x\in\Omega\,|\,\mathrm{dist}\,(x;\Gamma(t))<\delta\}$ is uniquely represented as

$$x=\gamma(t,y)+r\nu(t,y), \tag{2.2}$$

which gives us a new coodinate system (t,r,y). We denote by $J(t,r,y)$ Jacobian associated with (2.2). Namely,

$$J(t,r,y)=\prod_{i=1}^{N-1}(1+r\kappa_i(t,y))=:1+\sum_{i=1}^{N-1}H_i(t,y)r^i,$$

where $\kappa_i(t,y)$ $(i=1,\cdots,N-1)$ stand for the principal curvatures of $\Gamma(t)$ at $x=\gamma(t,y)$.

Let u^ϵ be a solution of (RD) for an appropreate initial condition:

$$\epsilon u_t^\epsilon(t,x)=\epsilon^2\Delta u^\epsilon(t,x)+f(u^\epsilon(t,x))-v^\epsilon(t),\quad t>0,\ x\in\Omega, \tag{2.3}$$

$$v^\epsilon(t)=\frac{1}{|\Omega|}\int_\Omega f(u^\epsilon(t,x))\,dx,\quad t>0. \tag{2.4}$$

We define an interface $\Gamma^\epsilon(t)$ as a level set of the solution u^ϵ to (RD). Since transition layers are expected to develop in regions $\{x\in\Omega\,|\,u^\epsilon(t,x)\approx h^0(v^*)\}$, we set (cf. Remark 2)

$$\Gamma^\epsilon(t):=\{x\in\Omega\,|\,u^\epsilon(t,x)=0\}. \tag{2.5}$$

On the other hand, $\Gamma^\epsilon(t)$ is also assumed to be expressed as a graph of a smooth function over the interface $\Gamma(t)$:

$$\Gamma^\epsilon(t)=\{x\in\Omega\,|\,x=\gamma(t,y)+\epsilon R^\epsilon(t,y)\nu(t,y),\ y\in\mathcal{M}\}. \tag{2.6}$$

R^ϵ, of course, is a priori unknown and is to be determined.

2.2 Outer expansion.

We separate the whole domain Ω into two components $\Omega^{\epsilon,\pm}(t)$ by the interface $\Gamma^\epsilon(t)$ such as $\Omega=\Omega^{\epsilon,-}(t)\cup\Gamma^\epsilon(t)\cup\Omega^{\epsilon,+}(t)$, and substitute the formal expansions

$$U^\epsilon(t,x)=U^{\epsilon,\pm}(t,x)=\sum_{j\geq 0}\epsilon^j U^{j,\pm}(t,x),\quad v^\epsilon(t)=\sum_{j\geq 0}\epsilon^j v^j(t) \tag{2.7}$$

into (2.3) in order to see the profile of solutions away from layer regions. Equating to zero the coefficient of each power of ϵ in the resulting equation, we obtain the following series of equations:

$$f(U^{0,\pm}) - v^0 = 0, \tag{2.8}$$

$$f'(U^{0,\pm})U^{j,\pm} = v^j + F_j^\pm, \qquad j \geq 1. \tag{2.9}$$

Here $F_j^\pm$ stand for functions depending on $U^{k,\pm}$ $(0 \leq k < j)$ only.

As the solution of (2.8), noting that **(A1)**, we choose

$$U^{0,\pm}(t,x) := h^\pm(v^0(t)). \tag{2.10}$$

Once we make this choice, $U^{j,\pm}$ $(j \geq 1)$ can be successively expressed by (2.9) as

$$U^{j,\pm}(t,x) = h_v^\pm(v^0(t))v^j(t) + V_j^\pm(t) \tag{2.11}$$

with $V_j^\pm$ being some functions depending on v^k $(0 \leq k < j)$. Therefore once v^j is known, $U^{j,\pm}$ are determined completely. v^j $(j \geq 0)$ will be determined later so that the C^1-matching conditions are satisfied (cf. subsection 2.5). We note, in particular, that the outer solution $U^\epsilon(t,x)$ is independent of x, and therefore is denoted simply as $U^\epsilon(t)$ in the sequel.

2.3 Inner expansion.

To deal with layer phenomena near $r = \epsilon R^\epsilon(t,y)$ (cf. (2.2), (2.6)), we use a stretched variable $z := \epsilon^{-1}[r - \epsilon R^\epsilon(t,y)]$ and recast our problem (2.3) in terms of (t,z,y):

$$\tilde{u}^\epsilon_{zz} + (\gamma_t \cdot \nu)\tilde{u}^\epsilon_z + f(\tilde{u}^\epsilon) + \epsilon R^\epsilon_t \tilde{u}^\epsilon_z - v^\epsilon + \mathcal{D}^\epsilon \tilde{u}^\epsilon = 0, \quad z \in (-\delta/\epsilon - R^\epsilon,\ \delta/\epsilon - R^\epsilon), \tag{2.12}$$

where $\mathcal{D}^\epsilon$ stands for a differential operator including R^ϵ.

We will seek an asymptotic solution to (2.12) of the form

$$\tilde{u}^\epsilon(t,z,y) = U^\epsilon(t,x)|_{x=\gamma(t,y)+(\epsilon z+\epsilon R^\epsilon(t,y))\nu(t,y)} + \phi^\epsilon(t,z,y) = U^\epsilon(t) + \phi^\epsilon(t,z,y), \tag{2.13}$$

i.e., we will determine ϕ^ϵ in such a way that $\tilde{u}^\epsilon$ in (2.13) asymptotically satisfies (2.12) for $z \in (-\infty,\infty)$. We substitute the formal expansions

$$R^\epsilon(t,y) = R^1(t,y) + \epsilon R^2(t,y) + \epsilon^2 R^3(t,y) + \cdots, \tag{2.14}$$

$$\begin{aligned} \tilde{u}^\epsilon(t,z,y) &= \tilde{u}^{\epsilon,\pm}(t,z,y) = U^{\epsilon,\pm}(t) + \phi^{\epsilon,\pm}(t,z,y) \\ &= \sum_{j\geq 0}\epsilon^j U^{j,\pm}(t) + \sum_{j\geq 0}\epsilon^j\phi^{j,\pm}(t,z,y) =: \sum_{j\geq 0}\epsilon^j\tilde{u}^{j,\pm}(t,z,y) \end{aligned} \tag{2.15}$$

together with the expansion for v^ϵ into (2.12) to obtain some series of equations for $\tilde{u}^{j,\pm}$ and $\tilde{\phi}^{j,\pm}$ in $\pm z \in (0,\infty)$. We now exibit equations for $\tilde{u}^{j,\pm}$ only:

$$\tilde{u}^{0,\pm}_{zz} + (\gamma_t \cdot \nu)\tilde{u}^{0,\pm}_z + f(\tilde{u}^{0,\pm}) - v^0 = 0, \tag{2.16}$$

$$\tilde{u}^{j,\pm}_{zz} + (\gamma_t \cdot \nu)\tilde{u}^{j,\pm}_z + f'(\tilde{u}^{0,\pm})\tilde{u}^{j,\pm} = v^j - R^j_t\tilde{u}^{0,\pm}_z + \mathcal{F}_j^\pm, \qquad j \geq 1. \tag{2.17}$$

Here $\mathcal{F}_j^\pm$ stand for functions depending on R^k, v^k, $\tilde{u}^{k,\pm}$ $(0 \leq k < j)$ with $R^0 := \gamma$.

We impose the following conditions:

- Boundary conditions at $z=0$ (cf. (2.5)):

$$\tilde{u}^{j,\pm}(t,0,y) = U^{j,\pm}(t) + \phi^{j,\pm}(t,0,y) = 0. \tag{2.18}$$

- Boundary conditions at $z=\pm\infty$ (outer-inner matching conditions):

$$\phi^{j,\pm}(t,z,y) \to 0 \ \text{ exponentially as } \ z \to \pm\infty. \tag{2.19}$$

- C^1-matching conditions at $z=0$:

$$\tilde{u}_z^{j,-}(t,0,y) = \tilde{u}_z^{j,+}(t,0,y). \tag{2.20}$$

2.4 Expansion of nonlocal term.

(2.4) is recast as follows:

$$\begin{aligned}
&\dot{U}^{\epsilon,-}|\Omega^-| + \dot{U}^{\epsilon,+}|\Omega^+| \\
&\quad = (\dot{U}^{\epsilon,+} - \dot{U}^{\epsilon,-}) \sum_{i\geq 0} \int_{\mathcal{M}} \frac{H_i(t,y)}{i+1}\Big(\epsilon R^\epsilon(t,y)\Big)^{i+1} dS_y^{\gamma(t,\cdot)} \\
&\qquad + \int_{\mathcal{M}} \int_{-\infty}^0 [\phi_{zz}^{\epsilon,-} + (\gamma_t\cdot\nu)\phi_z^{\epsilon,-} + \epsilon R_t^\epsilon \phi_z^{\epsilon,-} + \mathcal{D}^\epsilon \phi^{\epsilon,-}] J^\epsilon \, dz dS_y^{\gamma(t,\cdot)} \\
&\qquad + \int_{\mathcal{M}} \int_0^\infty [\phi_{zz}^{\epsilon,+} + (\gamma_t\cdot\nu)\phi_z^{\epsilon,+} + \epsilon R_t^\epsilon \phi_z^{\epsilon,+} + \mathcal{D}^\epsilon \phi^{\epsilon,+}] J^\epsilon \, dz dS_y^{\gamma(t,\cdot)} \\
&\qquad + O(\epsilon^{-1} e^{-\delta/\epsilon}).
\end{aligned} \tag{2.21}$$

Here $J^\epsilon(t,z,y) := J(t,r,y)|_{r=\epsilon z + \epsilon R^\epsilon(t,y)}$ and $dS_y^{\gamma(t,\cdot)}$ stands for the volume element on $\mathcal{M}$ induced from $dS_x^{\Gamma(t)}$, the surface element on $\Gamma(t)$ at x, by the embedding $\gamma(t,\cdot)$. These are denoted simply as dS_x and dS_y in the sequel.

We substitute the outer and inner expansions into (2.21) to obtain some series of equations:

$$\begin{aligned}
\dot{U}^{0,-}|\Omega^-| + \dot{U}^{0,+}|\Omega^+| \ &= \int_{\mathcal{M}} \int_{-\infty}^0 [\phi_{zz}^{0,-} + (\gamma_t\cdot\nu)\phi_z^{0,-}] \, dz dS_y \\
&\quad + \int_{\mathcal{M}} \int_0^\infty [\phi_{zz}^{0,+} + (\gamma_t\cdot\nu)\phi_z^{0,+}] \, dz dS_y,
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
&\dot{U}^{j,-}|\Omega^-| + \dot{U}^{j,+}|\Omega^+| \\
&\quad = (\dot{U}^{0,+} - \dot{U}^{0,-}) \int_{\mathcal{M}} R^j \, dS_y \\
&\qquad + \int_{\mathcal{M}} \int_{-\infty}^0 [\phi_{zz}^{0,-} + (\gamma_t\cdot\nu)\phi_z^{0,-}]\kappa R^j \, dz dS_y \\
&\qquad + \int_{\mathcal{M}} \int_0^\infty [\phi_{zz}^{0,+} + (\gamma_t\cdot\nu)\phi_z^{0,+}]\kappa R^j \, dz dS_y \\
&\qquad + \int_{\mathcal{M}} \int_{-\infty}^0 [\phi_{zz}^{j,-} + (\gamma_t\cdot\nu)\phi_z^{j,-} + R_t^j \phi_z^{0,-}] \, dz dS_y \\
&\qquad + \int_{\mathcal{M}} \int_0^\infty [\phi_{zz}^{j,+} + (\gamma_t\cdot\nu)\phi_z^{j,+} + R_t^j \phi_z^{0,+}] \, dz dS_y + \mathcal{I}^j, \quad j \geq 1.
\end{aligned} \tag{2.23}$$

Here $\kappa := \kappa_1 + \cdots + \kappa_{N-1}$ is the mean curvature of $\Gamma(t)$ at $x = \gamma(t,y)$, and $\mathcal{I}^j(t)$ stands for a function calculated by using functions R^k, $U^{k,\pm}$ and $\phi^{k,\pm}$ $(0 \le k < j)$.

2.5 C^1-matching.

We note that the following problem

$$(2.24) \qquad \begin{cases} Q_{zz} + cQ_z + f(Q) - v = 0, & z \in (-\infty, \infty), \\ Q(\pm\infty) = h^\pm(v), \quad Q(0) = 0, \end{cases}$$

has a unique solution pair $(Q(z;v), c(v))$ for each $v \in I^0$. Then (2.16) with (2.18)-(2.20) have unique solutions if and only if

$$(2.25) \qquad \gamma_t(t,y) \cdot \nu(t,y) = c(v^0(t)) \qquad v^0(t) \in I^0,$$

and solutions are given by

$$(2.26) \qquad \tilde{u}^{0,\pm}(t,z,y) = Q(z; v^0(t)), \quad \pm z \in (0,\infty).$$

Once (2.25) is satisfied and we have (2.26), we can successively show the existence and uniqueness of $\phi^{j,\pm}$ satisfying (2.19) for all $j \ge 0$.

As for $\tilde{u}^{j,\pm}$ $(j \ge 1)$, equations (2.17) with (2.18)-(2.20) have unique solutions if and only if a solvability condition of (2.17)

$$\int_{-\infty}^{\infty} e^{cz} Q_z (v^j - R_t^j Q_z + \mathcal{F}_j)\, dz = 0$$

is satisfied, which is equivalent to

$$(2.27) \qquad R_t^j(t,y) = c'(v^0(t))\, v^j(t) + \rho_j(t,y)$$

with ρ_j being a function calculated by using R^k, v^k and $\tilde{u}^k$ $(0 \le k < j)$. For instance, ρ_1 is given by

$$(2.28) \qquad \rho_1 = -\kappa + \frac{\int_{-\infty}^{\infty} e^{c(v^0)z} Q_z(z;v^0)\, Q_v(z;v^0)\, dz}{\int_{-\infty}^{\infty} e^{c(v^0)} [Q_z(z;v^0)]^2\, dz}\, \dot{v}^0.$$

On the other hand, (2.22) and (2.23) with (2.18)-(2.20) respectively yield

$$(2.29) \qquad \dot{v}^0(t) = \frac{h^+(v^0(t)) - h^-(v^0(t))}{h_v^-(v^0(t))|\Omega^-(t)| + h_v^+(v^0(t))|\Omega^+(t)|}\, c(v^0(t))\, |\Gamma(t)|,$$

$$(2.30) \qquad \dot{v}^j(t) = \int_{\mathcal{M}} a(t,y) R^j(t,y)\, dS_y + b(t)\, v^j(t) + \sigma_j(t).$$

Here a and b are some functions depending only on (Γ, v^0) given by

$$(2.31) \qquad a := \frac{[h^+(v^0) - h^-(v^0)]\, c(v^0)\, \kappa + [h_v^+(v^0) - h_v^-(v^0)]\, \dot{v}^0}{h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|},$$

$$(2.32)\qquad \begin{aligned} b \;:=\; & \frac{h^+(v^0)-h^-(v^0)}{h_v^-(v^0)|\Omega^-|+h_v^+(v^0)|\Omega^+|}\,c'(v^0)\,|\Gamma| \\ & +\frac{(h_v^+(v^0)-h_v^-(v^0))\,c(v^0)\,|\Gamma|-(h_{vv}^-(v^0)|\Omega^-|+h_{vv}^+(v^0)|\Omega^+|)\,\dot v^0}{h_v^-(v^0)|\Omega^-|+h_v^+(v^0)|\Omega^+|}, \end{aligned}$$

while σ_j stands for a function computed by employing R^k, v^k and $\phi^{k,\pm}$ $(0 \le k < j)$. For instance, σ_1 is given by

$$\begin{aligned} \sigma_1 = & -\frac{h^+(v^0)-h^-(v^0)}{h_v^-(v^0)|\Omega^-|+h_v^+(v^0)|\Omega^+|}\int_{\mathcal{M}}\kappa\,dS_y \\ & +\Big[h_v^-(v^0)|\Omega^-|+h_v^+(v^0)|\Omega^+|\Big]^{-1}\times\Big[\,c(v^0)\Big(\int_{-\infty}^{\infty} zQ_z(z;v^0)\,dz\Big)\int_{\mathcal{M}}\kappa\,dS_y \\ & \quad -\frac{d}{dt}\Big(h_v^-(v^0)\dot v^0\Big)|\Omega^-|-\frac{d}{dt}\Big(h_v^+(v^0)\dot v^0\Big)|\Omega^+| \\ & \quad -\Big(\int_{-\infty}^{0}(Q_v(z;v^0)-h_v^-(v^0))\,dz+\int_0^{\infty}(Q_v(z;v^0)-h_v^+(v^0))\,dz\Big)\,\dot v^0\,|\Gamma| \\ & \quad +\Big(h_v^+(v^0)^2-h_v^-(v^0)^2\Big)\,c(v^0)\,(\dot v^0)^2\,|\Gamma| \\ & \quad +\Big(h^+(v^0)-h^-(v^0)\Big)\frac{\int_{-\infty}^{\infty}e^{c(v^0)z}Q_z(z;v^0)Q_v(z;v^0)\,dz}{\int_{-\infty}^{\infty}e^{c(v^0)z}[Q_z(z;v^0)]^2\,dz}\,\dot v^0\,|\Gamma|\,\Big]. \end{aligned}$$

We finally arrived at the following interface equations:

$$(\mathrm{IE}^0)\qquad \gamma_t\cdot\nu = c(v^0),\quad \dot v^0 = \frac{h^+(v^0)-h^-(v^0)}{h_v^-(v^0)|\Omega^-|+h_v^+(v^0)|\Omega^+|}\,c(v^0)\,|\Gamma|,$$

$$(\mathrm{IE}^j)\qquad R_t^j = c'(v^0)\,v^j+\rho_j,\quad \dot v^j = \int_{\mathcal{M}} a\,R^j\,dS_y + b\,v^j+\sigma_j,\qquad j\ge 1.$$

3 Analysis of interface equations.

We are now ready to study the interface equations. Let us begin with the 0-th order equation (IE^0).

3.1 0-th order equation.

The equation is as follows:

$$(\mathrm{IE}^0\text{-a})\qquad \mathbf{v}(x;\Gamma(t)) = c(v(t)),\qquad t>0,\ x\in\Gamma(t),$$

$$(\mathrm{IE}^0\text{-b})\qquad \dot v(t) = \frac{h^+(v(t))-h^-(v(t))}{h_v^-(v(t))|\Omega^-(t)|+h_v^+(v(t))|\Omega^+(t)|}\,c(v(t))\,|\Gamma(t)|,\qquad t>0,$$

$$(\mathrm{IE}^0\text{-c})\qquad \Gamma(0)=\Gamma_0,\quad v(0)=v_0.$$

Here $\mathbf{v}(x;\Gamma(t)) := \gamma_t(t,y)\cdot\nu(t,y)$ is the normal velocity of $\Gamma(t)$ at $x=\gamma(t,y)$. We note that the superscript '0' in $v^0(t)$ has been suppressed.

It immediately turns out, due to (IE^0-a), that the normal speed is independent of the position $x\in\Gamma(t)$ and is regulated by the (0-th order) nonlocal term v. Thanks to the identity

$$\frac{d}{dt}|\Omega^-(t)| = -\frac{d}{dt}|\Omega^+(t)| = \int_{\Gamma(t)} \mathbf{v}(x;\Gamma(t))\,dS_x, \tag{3.1}$$

the interface equation (IE^0) implies

$$h^-(v(t))\frac{|\Omega^-(t)|}{|\Omega|} + h^+(v(t))\frac{|\Omega^+(t)|}{|\Omega|} \equiv m_0, \qquad t\geq 0, \tag{3.2}$$

where $m_0 = m_0(\Gamma_0, v_0)$ is given by

$$m_0 := h^-(v_0)\frac{|\Omega_0^-|}{|\Omega|} + h^+(v_0)\frac{|\Omega_0^+|}{|\Omega|} \tag{3.3}$$

with $\Omega_0^\pm$ being initial bulk regions such as $\Omega = \Omega_0^-\cup\Gamma_0\cup\Omega_0^+$. We note that (3.2) corresponds to (PP) for (RD) as $\epsilon\to 0$ (cf. (2.10)).

We recast (IE^0) as a system of ordinary differential equations after the manner of Sakamoto [11]. For a given initial interface Γ_0 we express $\Gamma(t)$ as the graph of a function $r(t,y)$ over Γ_0: $\gamma(t,y) = \gamma(0,y) + r(t,y)\nu(0,y)$. Then some elementary calculations yield $\nu(t,y)\equiv\nu(0,y)$ and $r(t,y)\equiv r(t)$, and therefore (IE^0-a) is recast as $\dot r(t) = c(v(t))$. On the other hand, the surface area of an interface $\{x\in\Omega \mid x = \gamma(0,y) + r\nu(0,y),\ y\in\mathcal{M}\}$ is given by

$$g(r) := \int_{\mathcal{M}} J(0,r,y)\,dS_y^0 = |\Gamma_0| + \sum_{i=1}^{N-1}\Big(\int_{\mathcal{M}} H_i(0,y)\,dS_y^0\Big)r^j, \quad dS_y^0 := dS_y^{\gamma(0,\cdot)},$$

so we have $|\Gamma(t)| = g(r(t))$. Moreover, (3.2) together with $|\Omega^-(t)|+|\Omega^+(t)|\equiv|\Omega|$ implies that the volume of the bulk regions are represented in terms of v as

$$|\Omega^-| = \frac{h^+(v)-m_0}{h^+(v)-h^-(v)}|\Omega|, \qquad |\Omega^-| = \frac{m_0-h^-(v)}{h^+(v)-h^-(v)}|\Omega|, \tag{3.4}$$

from which the first factor in the right hand side of (IE^0-b) is rewritten as $h(v(t))$ with

$$h(v) = h(v;v_0) := \frac{1}{|\Omega|}\frac{[h^+(v)-h^-(v)]^2}{h_v^-(v)[h^+(v)-m_0] + h_v^+(v)[m_0-h^-(v)]}. \tag{3.5}$$

In particular, if the initial pair (Γ_0, v_0) is chosen so that $m_0\in(\underline{u},\overline{u})$, it follows that $|\Omega^\pm|>0$ in (3.4) and therefore we have $h(v)<0$ for all $v\in I^0$ (cf. **(A1)**, **(A2)**). Thus the interface equation (IE^0) are equivalent to the following initial value problem:

$$\begin{cases} \dot r = c(v), \\ \dot v = h(v)\,c(v)\,g(r), \\ r(0) = 0, \quad v(0) = v_0. \end{cases} \tag{ODE^0}$$

By virtue of reformulation above and an equivalent expression of $c(v)$

$$c(v) = -\frac{\mathcal{S}(v)}{\int_{-\infty}^{\infty} [Q_z(z;v)]^2\, dz}, \tag{3.6}$$

the interface dynamics are summerized as follows:

- $v \in (v^*, \overline{v}) \quad \Longrightarrow \quad \dot{r} > 0, \quad \dot{v} < 0;$
  the interface $\Gamma(t)$ evolves in such a way that the bulk region $\Omega^-(t)$ grows uniformly.
- $v \in (\underline{v}, v^*) \quad \Longrightarrow \quad \dot{r} < 0, \quad \dot{v} > 0;$
  the interface $\Gamma(t)$ evolves in such a way that the bulk region $\Omega^-(t)$ shrinks uniformly.
- $v = v^* \quad \Longrightarrow \quad \dot{r} = 0, \quad \dot{v} = 0;$
  the interface $\Gamma(t)$ does not evolve.

We also obtain the following

Theorem 3 (Unique existence of solutions). *Let Γ_0 be a smooth initial interface, and a pair (Γ_0, v_0) is assumed to satisfy $v_0 \in I^0$ and $m_0 \in (\underline{u}, \overline{u})$. Then the following statements hold true:*

(1) *There exists a constant $T > 0$ such that* (IE^0) *has a unique smooth solution pair (Γ, v) on a time interval $[0, T]$.*

(2) *If in addition v_0 is sufficiently close to v^*, then the unique solution (Γ, v) in (1) exists globally in time.*

Proof. (2) immediately follows from the existence of a constant $R > 0$ such that r-component $r(\cdot)$ of the solution to (ODE^0) remains in a neighborhood $(-R, R)$ while the corresponding interface $\Gamma(\cdot) = \{x \in \Omega \,|\, x = \gamma(0, y) + r(\cdot)\nu(0, y),\ y \in \mathcal{M}\}$ is smooth for all $|r| < R$ when we choose $v_0 \approx v^*$. □

Theorem 4 (Stability of equilibrium solutions). *Suppose that a pair (Γ_0, v_0) is as in Theorem 3. Then the following statements hold true:*

(1) *(Γ_0, v_0) is an equilibrium solution of* (IE^0) *if and only if $v_0 = v^*$.*

(2) *The equilibrium solution (Γ_0, v^*) is asymptotically stable relative to* (ODE^0).

Proof. (2) We linearize (ODE^0) around the corresponding equilibrium solution $(0, v^*)$ to obtain the eigenvalues 0 and $h(v^*)\, c'(v^*)\, |\Gamma_0| < 0$. □

For each $v \in I^0$, the nonlinear term $f(u) - v$ defines a new double-well potential $\mathcal{W}(u; v)$ with two wells located at $u = h^{\pm}(v)$. Moreover, the potential difference is related to $\mathcal{S}(v)$ and $c(v)$ as follows:

$$\mathcal{W}(h^+(v); v) - \mathcal{W}(h^-(v); v) = -\mathcal{S}(v) = c(v) \int_{-\infty}^{\infty} [Q_z(z; v)]^2\, dz.$$

Hence it turns out that the 0-th order nonlocal effect equalizes the potential of two wells no matter how the initial state is.

3.2 Higher order equations.

The j-th ($j \geq 1$) order equations are as follows:

$$(\mathrm{IE}^j\text{-a}) \qquad R^j_t(t,y) = c'(v^0(t))\, v^j(t) + \rho_j(t,y), \qquad t>0,\ y \in \mathcal{M},$$

$$(\mathrm{IE}^j\text{-b}) \qquad \dot{v}^j(t) = \int_{\mathcal{M}} a(t,y) R^j(t,y)\, dS_y + b(t)\, v^j(t) + \sigma_j(t), \qquad t>0,$$

$$(\mathrm{IE}^j\text{-c}) \qquad R^j(0,y) = R^j(y), \quad v^j(0) = v^j_0.$$

Recall that a and b are functions depending only on the solution (Γ, v^0) to (IE^0) (cf. (2.31), (2.32)), while ρ_j and σ_j are some functions which can be calculated by using functions with index k ($0 \leq k < j$) in outer and inner expansions.

Each equation (IE^j) can be recast as a system of linear non-homogeneous ordinary differential equations. Indeed, by employing a function r^j given by

$$r^j(t) := R^j(t,y) - R^j(y) - \int_0^t \rho_j(s,y)\, ds,$$

$(\mathrm{IE}^j\text{-a})$ and $(\mathrm{IE}^j\text{-b})$ are respectively expressed as

$$\dot{r}^j(t) = c'(v^0(t))\, v^j(t),$$

$$\begin{aligned} \dot{v}^j(t) &= \Big(\int_{\mathcal{M}} a(t,y)\, dS_y\Big) r^j(t) + b(t)\, v^j(t) \\ &\quad + \int_{\mathcal{M}} a(t,y)\Big(R^j(y) + \int_0^t \rho_j(s,y)\, ds\Big)\, dS_y + \sigma_j(t), \end{aligned}$$

from which we obtain an initial value problem of the form

$$(\mathrm{ODE}^j) \qquad \begin{cases} \dot{r}^j(t) = \qquad\qquad B(t)\, v^j(t), \\ \dot{v}^j(t) = C(t)\, r^j(t) + D(t)\, v^j(t) + E_j(t), \\ r^j(0) = 0, \quad v^j(0) = v^j_0. \end{cases}$$

Due to this reformulation, we have the following

Theorem 5 (Unique existence of solutions). *Once the initial pair $(R^j(y), v^j_0)$ is given, the equations (IE^j) ($j \geq 1$) are successively solvable on a finite time interval $[0,T]$.*

In particular, we can construct a smooth approximate solution u^ϵ_A of (RD) in the sense that

$$\Big\| \epsilon \frac{\partial u^\epsilon_A}{\partial t} - \epsilon^2 \Delta u^\epsilon_A - f(u^\epsilon_A) + \frac{1}{|\Omega|}\int_\Omega f(u^\epsilon_A(\cdot,x))\, dx \Big\|_{L^\infty([0,T]\times\Omega)} = O(\epsilon^{K+1}),$$

$$\frac{\partial u^\epsilon_A}{\partial \mathbf{n}} = 0, \qquad (t,x) \in [0,T] \times \partial\Omega,$$

by means of unique solutions (Γ, v^0) and (R^j, v^j) of (IE^j) for $0 \leq j \leq K$.

As the solution $v^0(t)$ approaches the equilibrium state v^*, the 0-th order equation (IE^0) becomes powerless to approximate the layer dynamics. In this case, we must move our attention to the equation (IE^1) for (R^1, v^1) in order to capture the further dynamics of layers. An investigation in such a direction will be our future work.

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